Counting H-free graphs for bipartite H

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Outline

- Background & definitions
- Conjecture, progress, & proof methods
- Our result
- Proof sketch & applications

Background & Definitions

Background & Definitions

- The **extremal number** $\mathrm{ex}(n, H)$ for a graph H is the maximum possible number of edges in a graph G on n vertices that does not contain a H as a subgraph.
- Call such a graph **H-free**.

Classical result of Turán (1941) and Erdős-Stone (1946):

Erdős-Stone Theorem

$$\mathrm{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2)$$

- This gives asymptotic behavior of $\mathrm{ex}(n, H)$ when $\chi(H) \geq 3$, but what about bipartite graphs?
- **Answer:** Very tricky!
- See survey of Füredi and Simonovits, 2013 (97 pages!)

Background & Definitions

- **Closely related problem:** count H-free graphs.
- Explicitly, find $|\mathcal{F}_n(H)|$, the number of (labeled) graphs on n vertices that do not contain H as a subgraph.
- How is this related to finding $\mathrm{ex}(n, H)$?

Background & Definitions

Trivial bounds

- **Lower bound:** $|\mathcal{F}_n(H)| \geq 2^{\mathbf{ex}(n,H)}$
- **Upper bound:** $|\mathcal{F}_n(H)| \leq \sum_{i=0}^{\mathrm{ex}(n,H)} \binom{\binom{n}{2}}{i} = 2^{O(ex(n,H)\log(n))}$
- **Question:** How to eliminate $log(n)$ factor?

Better bounds

- In general, $|\mathcal{F}_n(H)| = 2^{\mathbf{ex}(n,H)+o(n^2)}$, proved by Erdős, Frankl, and Rödl in 1986.
- If $\chi(H) \geq 3$, then this means $|\mathcal{F}_n(H)| = 2^{(1+o(1))\,\mathbf{ex}(n,H)}$
- But if H is a forest, $|\mathcal{F}_n(H)| = 2^{\Theta(\mathbf{ex}(n,H)\,\log(n))}$

Conjecture (Erdős, Frankl, and Rödl, 1986):

For any H containing a cycle,

$$|\mathcal{F}_n(H)| = 2^{(1+o(1))\,\mathrm{ex}(n,H)}$$

- **False!**
- **Counterexample:** $|\mathcal{F}_n(C_6)| \geq 2^{(1+c)\,\mathrm{ex}(n,H)}$ for some $c > 0$; Morris and Saxton (2016).

The Problem

New Conjecture

For any H containing a cycle,

$$|\mathcal{F}_n(H)| = 2^{O(\mathrm{ex}(n,H))}$$

Progress

- Known for C_4, C_6, and C_{10}.
- Known for $K_{2,t}$, $K_{3,t}$, and $K_{s,t}$ with $t > (s-1)!$.
- ”Almost” known for some others - e.g. $|\mathcal{F}_n(C_{2\ell})| = 2^{O(n^{1+1/\ell})}$. Known that $\text{ex}(n, C_{2\ell}) = O(n^{1+1/\ell})$, conjectured to be sharp.
- Known for k-uniform hypergraphs with $\chi(H) > k$ (non-degenerate case).

Methods

- **Main technique:** hypergraph containers (Balogh, Morris, and Samotij, 2015; Saxton and Thomason, 2015)
- Gives a way to count independent sets in hypergraphs.

- **Application:** create hypergraph $\mathcal{Z}$ whose vertices are the edges of K_n and whose edges are all copies of H in K_n.
- Then H-free graphs on n vertices correspond to independent sets in $\mathcal{Z}$.

Containers Method

- **Broad strokes**: for a hypergraph $\mathcal{Z}$ satisfying certain "niceness" properties, there exists a family of containers $\mathcal{C} \subseteq \mathcal{P}(V(\mathcal{Z}))$ so that each independent set in $\mathcal{Z}$ is contained in some $C \in \mathcal{C}$.
- So $|\mathcal{F}_n(H)| \leq (\#\text{ containers}) \cdot 2^{\text{size of max container}}$

Containers

- **”Niceness”**: In general, in any graph with more than $\mathrm{ex}(n, H)$ edges, need to prove there are ”many” and ”well-distributed” copies of H (a **supersaturation** condition) in order to apply containers.
- Supersaturation results often *very* hard to prove
- *”If only he had used his genius for niceness instead of evil”*

A New Hope

- **Question:** Possible to prove supersaturation without knowing $\mathrm{ex}(n, H)$?
- **Answer:** Maybe! Paper by Balogh, Liu, and Sharifzadeh (2016) counting k-arithmetic progression free subsets of $[n]$.
- Sample smaller set of numbers, show they induce many k-APs, end up having to bound ratio of $\frac{\mathrm{ex}(m)}{\mathrm{ex}(n)}$ for $m < n$.

Our Contribution

Our result

Main Theorem

If H is any graph containing a cycle, and $\mathrm{ex}(n, H) = O(n^{\alpha})$ for some $\alpha \in (1, 2)$, then

$$|\mathcal{F}_n(H)| = 2^{O(n^{\alpha})}$$

In particular, if $\mathrm{ex}(n, H) = \Theta(n^{\alpha})$, then $|\mathcal{F}_n(H)| = 2^{O(\mathrm{ex}(n,H))}$.

Proof Ideas

- **First:** Inductive application of containers. Developed by Morris and Saxton in paper on $C_{2\ell}$-free graphs (2016).
- **Second:** Prove supersaturation result by bounding number of copies of H in small random subgraphs.

Notation

Notation

- $\gamma > 1$ is a constant depending on H
- v_H = # vertices of H
- e_H = # edges of H

Supersaturation Condition

Supersaturation Condition

Let k be any constant depending only on H. If for every graph G on n vertices with $\boldsymbol{m = \gamma^t \cdot k \cdot n^\alpha}$ edges, there exists a subset $\mathcal{Z}$ of all copies of H in G so that

$$\Delta_\ell(\mathcal{Z}) \leq \left(\frac{n^\alpha}{m(t+1)^3}\right)^{\ell-1} \cdot \frac{|\mathcal{Z}|}{m}$$

for all $\ell \in \{1, \ldots, e_H\}$ then $|\mathcal{F}_n(H)| = 2^{O(n^\alpha)}$.

$\Delta_\ell(\mathcal{Z})$ = maximum number of copies of H in $\mathcal{Z}$ that contain any subset of ℓ edges in G.

Proof of Supersaturation

For $\ell = e_H$, the condition

$$\Delta_\ell(\mathcal{Z}) \leq \left(\frac{n^\alpha}{m(t+1)^3}\right)^{\ell-1} \cdot \frac{|\mathcal{Z}|}{m}$$

reduces to

$$|\mathcal{Z}| \geq \boldsymbol{(\gamma^t \cdot k(t+1)^3)^{e_H-1} \cdot m}.$$

Show only this case here - gives basic idea of the proof.

Proof of Supersaturation

- **Goal:** given graph G with $m = \gamma^t \cdot k \cdot n^\alpha$ edges, want to show there at least $(\gamma^t \cdot k(t+1)^3)^{e_H - 1} \cdot m$ copies of H
- **Strategy:** show that random small subgraph of G gives many copies of H.

Proof of Supersaturation

Notation

- R = uniformly random set of pn vertices in G
- $p \in (0, 1)$, yet to be chosen
- X = number of copies of H in induced subgraph $G[R]$ (random variable)
- Z = total number of copies of H in G (what we're trying to bound)

Bounds

- $X \geq e(G[R]) - \mathrm{ex}(pn, H)$
- So $\mathbb{E}[X] \geq \mathbb{E}[e(G[R])] - \mathrm{ex}(pn, H)$

And

- $\mathbb{E}[X] = Z \cdot \binom{n-v_H}{pn-v_H} / \binom{n}{pn} \approx Z \cdot p^{v_H}$
- $\mathbb{E}[e(G[R])] = m \cdot \binom{n-2}{pn-2} / \binom{n}{pn} \approx m \cdot p^2$

Solve to get:

$$Z \geq (mp^2 - \mathrm{ex}(pn, H))p^{-v_H}.$$

Proof of Supersaturation

Goal: Show that random subgraph $G[R]$ of correct size pn has many copies of H. Use this to bound total number Z of copies.

Show:

$$Z \geq (mp^2 - \text{ex}(pn, H))p^{-v_H} \geq (\gamma^t \cdot k(t+1)^3)^{e_H - 1} \cdot m$$

Proof of Supersaturation

Approach

- Do some algebra to get upper and lower bonds on p
- Along the way, use fact that $\frac{\mathrm{ex}(pn,H)}{n^\alpha} \leq \frac{p^\alpha n^\alpha}{n^\alpha} = \boldsymbol{p^\alpha}$
- End up with upper bound $\geq$ lower bound if and only if

$$\frac{e_H - 1}{v_H - 2} < \frac{1}{2 - \alpha}$$

Proof of Supersaturation

$$\frac{e_H - 1}{v_H - 2} < \frac{1}{2 - \alpha} ?$$

Definition: $m_2(H) = \max\left\{\frac{e(F)-1}{v(F)-2} : F \subseteq H \text{ with } e(F) > 1\right\}.$

Bohman and Keevash, 2009

For any H containing a cycle,

$$\mathrm{ex}(n, H) \geq n^{2-1/m_2(H)} \cdot \log(n)^{\frac{1}{e_H - 1}}.$$

Since $n^\alpha \geq \mathrm{ex}(n, H)$,

$$n^\alpha \geq n^{2-1/m_2(H)} \cdot \log(n)^{\frac{1}{e_H - 1}}.$$

Proof of Supersaturation

$$\frac{e_H - 1}{v_H - 2} < \frac{1}{2 - \alpha} \; ?$$

Know:

$$n^{\alpha - 2 + 1/m_2(H)} > \log(n)^{\frac{1}{e_H - 1}}$$

So

$$\alpha - 2 + 1/m_2(H) > 0$$

And in particular,

$$\alpha - 2 + \frac{v_H - 2}{e_H - 1} > 0$$

Summary

Main Ideas in Proof of Supersaturation

- Probabilistic method: show that random subgraph $G[R]$ of correct size has many copies of H.
- Use assumption on growth rate of $\mathrm{ex}(n, H)$ to bound ratio $\frac{\mathrm{ex}(pn,H)}{n^{\alpha}}$.
- Use bound on $\mathrm{ex}(n, H)$ in terms of 2-density $m_2(H)$ to show there is a gap between upper and lower bounds.

Applications

Reproving Old Results

- Reproves non-degenerate case (where $\chi(H) \geq 3$)
- Reproves $|\mathcal{F}_n(H)| = 2^{O(\mathrm{ex}(n,H))}$ for C_4, C_6, and C_{10}, as well as $K_{2,t}$, $K_{3,t}$, and $K_{s,t}$ with $t > (s-1)!$.
- Reproves $|\mathcal{F}_n(C_{2\ell})| = 2^{O(n^{1+1/\ell})}$ - result of Morris and Saxton (2016)
- Hypergraphs: reproves recent result of Balogh, Nayaranan, and Skokan (2017) for linear cycles: $|\mathcal{F}_n(C_k^r)| = 2^{O(\mathrm{ex}(n,C_k^r))}$
- The list goes on!

Infinite Sequences

If there is a constant $\varepsilon > 0$ such that $\text{ex}(n, H) = \Omega(n^{2-1/m_2(H)+\varepsilon})$, then there exist an infinite sequence $\{n_i\} \subseteq \mathbb{N}$ and a constant $C > 0$ such that

$$|\mathcal{F}_n(H)| \leq 2^{C \cdot \text{ex}(n,H)}$$

for all i.

In particular, this holds for all even cycles, $C_{2\ell}$. (Lubotzky, Phillips, and Sarnak, 1988).

Questions?